Technical Notes

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Calculations of Viscous Flow with Separation Using Newton's Method and Direct Solver

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Introduction

PRACTICAL application of direct solution techniques for fluid flow problems has improved significantly during recent years.[1-5] First, these direct solvers have become more efficient. Second, computers have become faster and their memories have grown in size almost exponentially. Currently, memories of even small computers are large enough to solve moderately large two-dimensional problems using a direct solver. The purpose of this study is to analyze the performance of fully implicit solution techniques that are based on Newton's method and direct solvers for banded matrices.[6] Several formulations of the two-dimensional steady Navier-Stokes equations are solved using this technique, and comparisons are made on the basis of CPU time, memory requirements, and solution accuracy for two separated flow problems. The first problem is an external flow with an adverse pressure gradient, while the second problem is a laminar flow in a diffuser. Three formulations are discussed based on the primitive variable, the stream-function vorticity, and the biharmonic equations. Previously, the direct solution technique was compared with the iterative solution technique for the stream-function vorticity formulation and found to be comparable in terms of CPU time.[7]

Primitive Variable Formulation

The equations of motion for the laminar incompressible steady two-dimensional flow in the conservative form and Cartesian coordinates are

$$(u^2)_x + (uv)_y = -p_x + \frac{1}{R}\nabla^2 u \tag{1}$$

$$(uv)_x + (v^2)_y = -p_y + \frac{1}{R}\nabla^2 v \tag{2}$$

$$u_x + v_y = 0 \tag{3}$$

The finite-difference form of the two momentum equations and the continuity equation are solved on a staggered grid. All the terms of the three equations are centrally differenced, and the nonlinear terms in the momentum equations are linearized using Newton's method. The x-momentum equation is centered around u at point $(i - \frac{1}{2}, j)$, the y-momentum equation is centered around v at point $(i, j + \frac{1}{2})$ and the continuity equation is centered around the pressure at point (i,j). Consequently, $u_{i,j}$, $v_{i,j}$ and $p_{i,j}$ are defined at points $(i - \frac{1}{2}, j)$, $(i, j + \frac{1}{2})$ and (i,j), respectively. For the external flow, the boundary conditions at the wall are

$$\delta v_{i,1} = 0 \tag{4}$$

$$\delta u_{i,1} + \delta u_{i,2} = 0 \tag{5}$$

At the upper boundary of the computational domain the velocity component u is known and pressure is determined by Bernoulli's equation (vorticity is zero)

$$\delta u_{i,J} = 0 \tag{6}$$

$$\delta p_{i,J} + \delta v_{i,J}(v_{i,J} + v_{i,J-1})/4 + \delta v_{i,J-1}(v_{i,J} + v_{i,J-1})/4$$
$$= 1.5 - p_{i,J} - [(u_{i,J} + u_{i+1,J})^2 + (v_{i,J} + v_{i,J-1})^2]/8 \tag{7}$$

At the inflow and at the outflow boundary, the flow is assumed to be governed by the boundary-layer equations. Consequently, the normal pressure gradient is zero and the pressure is known from the edge condition

$$\delta u_{1,j} = 0 \tag{8}$$

$$\delta p_{i,j+1} - \delta p_{i,j} = p_{i,j} - p_{i,j+1} \qquad \text{for } i = 1 \text{ and } i = I \tag{9}$$

In addition, the $(u^2)_x$-term is backward differenced and the u_{xx}-term is dropped at the last streamwise station. The resulting set of algebraic equations, including the boundary conditions, is solved for δu, δv, and δp with the direct solver. Iterations are required because of the nonlinear terms in the momentum equations. For the primitive variable formulation, the total storage requirement [including the matrices u, v, and p with the dimensions (M,N)] is estimated to be $3MN(9N + 14)$. Here, M and N represent the total number of grid points in the x- and y-direction $(N<M)$, respectively.

Stream-Function Vorticity Formulation

The Navier-Stokes equations in the stream-function vorticity form are

$$(\psi_y \omega)_x - (\psi_x \omega)_y = \frac{1}{R}\nabla^2\omega \tag{10}$$

$$-\omega = \nabla^2\psi \tag{11}$$

The finite-difference form of the equations is obtained using centered differences for all the terms and Newton's linearization. The boundary conditions for the external flow problem at the wall and the upper boundary are

$$\delta\psi_{i,1} = 0 \tag{12}$$

$$\frac{2}{(\Delta y)^2}(\delta\psi_{i,2} - \delta\psi_{i,1}) + \delta\omega_{i,1} = \frac{-2}{(\Delta y)^2}(\psi_{i,2} - \psi_{i,1}) - \omega_{i,1} \tag{13}$$

$$\delta\omega_{i,J} = 0 \tag{14}$$

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$$\delta\omega_{i,J} + \frac{1}{(\Delta x)^2}(\delta\psi_{i+1,J} - 2\delta\psi_{i,J} + \delta\psi_{i-1,J}) + \frac{2}{(\Delta y)^2}(\delta\psi_{i,J-1} - \delta\psi_{i,J}) = -\omega_{i,J} - \frac{1}{(\Delta x)^2}(\psi_{i+1,J} - 2\psi_{i,J} + \psi_{i-1,J}) - \frac{2}{(\Delta y)^2}(\psi_{i,J-1} - \psi_{i,J}) - 2U_e\Delta y \tag{15}$$

At the inflow boundary both the stream-function and the vorticity distributions are prescribed, and at the outflow boundary both the ψ_{xx} term and the ω_{xx} term are dropped. In addition, the $(u\omega)_x$ term and ψ_x are backward differenced at the last station. This coupled set of equations, together with the boundary conditions, are solved simultaneously for $\delta\psi$ and $\delta\omega$. Iterations are required to deal with the nonlinearity of the vorticity equation. The total required memory is estimated to be $2\,MN\,(6N + 13)$.

Biharmonic Formulation

The biharmonic formulation of the two-dimensional Navier-Stokes equations is simply obtained by eliminating ω from the stream-function vorticity formulation

$$(\psi_y \nabla^2\psi)_x - (\psi_x \nabla^2\psi)_y = \frac{1}{R}\nabla^4\psi \tag{16}$$

This fourth-order differential equation is discretized using centered differences for all terms and Newton's method is applied to linearize the resulting algebraic expression. Two imaginary grid-point locations are added; the first $(i,0)$ is located just outside the lower boundary and the second $(i,J+1)$ is located just outside the upper boundary. Consequently, the conditions at the solid wall are provided by Eq. (12) and:

$$\delta\psi_{i,2} - \delta\psi_{i,0} = \psi_{i,0} - \psi_{i,2} \tag{17}$$

The boundary conditions at the upper boundary for the external flow problem are

$$\frac{1}{(\Delta x)^2}(\delta\psi_{i+1,J} - 2\delta\psi_{i,J} + \delta\psi_{i-1,J}) + \frac{1}{(\Delta y)^2}(\delta\psi_{i,J+1} - 2\delta\psi_{i,J} + \delta\psi_{i,J-1}) = -\frac{1}{(\Delta x)^2}(\psi_{i+1,J} - 2\psi_{i,J} + \psi_{i-1,J}) - \frac{1}{(\Delta y)^2}(\psi_{i,J+1} - 2\psi_{i,J} + \psi_{i,J-1}) \tag{18}$$

$$\delta\psi_{i,J+1} - \delta\psi_{i,J-1} = 2U_e\Delta y - \psi_{i,J+1} + \psi_{i,J-1} \tag{19}$$

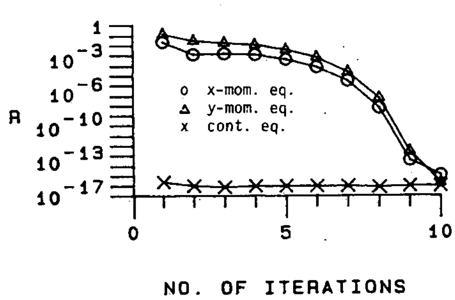

Fig. 1 Maximum absolute residuals of primitive variable equations for flat plate problem (61 × 31 grid).

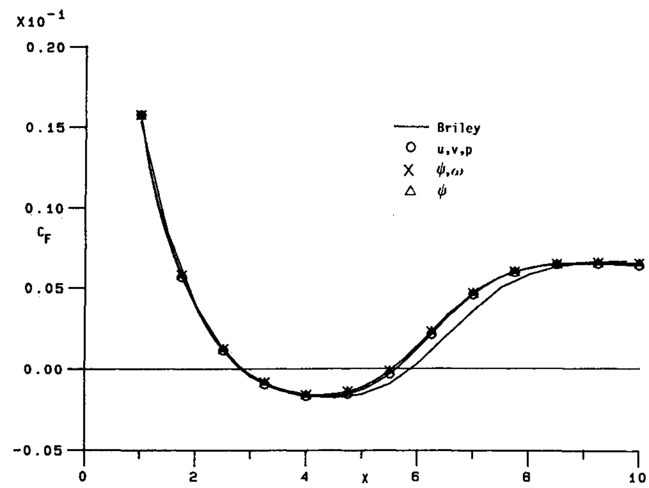

Fig. 2 Skin-friction distribution along flat plate for all three formulations. Uniform 61 × 31 grid for u,v,p and ψ,ω formulations. Uniform 61 × 33 grid for ψ formulation.

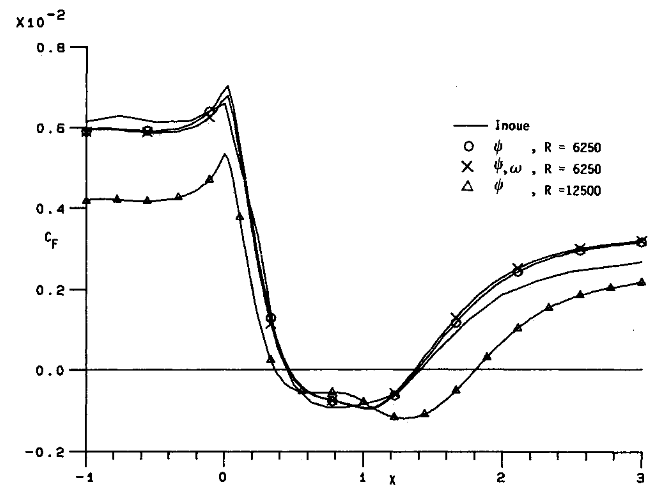

Fig. 3 Skin-friction distribution along diffuser wall. Uniform grid in streamwise direction; 91 grid point for $R = 6250$ and 181 for $R = 12500$. Stretched grid in normal direction; 31 grid points for ψ,ω and 33 for ψ formulation.

At the inflow boundary the ψ_{xx} term is dropped and the stream-function profile is specified. At the outflow boundary the ψ_{xx} term and the $(\nabla^2\psi)_{xx}$ term are both neglected, thus obtaining the boundary-layer equation in terms of the stream-function, and the terms $(\psi_y\psi_{yy})_x$ and ψ_x are backward differenced. The system of equations is solved simultaneously and several iterations are required to obtain a converged solution. For a $M \times N$ problem the estimated storage requirement is $MN\,(6N + 5)$.

Results and Discussion

First, the model problem of a separated flow on a flat plate, introduced by Briley,[8] is analyzed. An adverse pressure gradient prescribed along the upper boundary results in a separated flow region. The computational domain extends from $x = 1.0$ to $x = 10.0$ and from $y = 0$ to $y = 0.75$. The Reynolds number based on the free-stream velocity and the reference length $(x = 1.0)$ is $R = 1044$. The initial conditions for the entire flow field are the inflow boundary conditions. In Fig. 1, the convergence history for the primitive variable form of the equations is shown. The residuals for the two momentum equations reduce quadratically with the number of iterations. The linear continuity equation is always satisfied to machine accuracy. Convergence is obtained after seven iterations (independently of the grid size). The solution is considered converged when the maximum absolute residual is one order of magnitude smaller than the truncation error. The convergence history for the stream-function vorticity formulation is similar. The residual of the linear Poisson equation for the stream function is always of machine accuracy and the residual of the vorticity transport equation reduces quadratically; convergence is obtained after six iterations. For the biharmonic equation, convergence takes six iterations and is also quadratic. The solutions in terms of the wall skin-friction distribution are presented in Fig. 2. The results for the three formulations are in excellent agreement and correlate well with the skin-friction distribution presented by Briley.[8]

Next, the model problem of a separated flow in a symmetrical diffuser, introduced by Inoue,[9] is examined. The diffuser problem is solved using the stream-function vorticity and the biharmonic formulation. The inlet and outlet boundary of the diffuser are at $x = -1.0$ and $x = 3.0$, respectively. The shape factor of the diffuser wall $A = -0.08^9$. The centerline is located at $y = 1.0$ and the Reynolds number based on this reference length and the free-stream velocity is $R = 6250$. The inflow conditions provide the initial conditions for the entire flowfield. Convergence is quadratic and machine zero is reached in 6-7 iterations. The solutions in terms of the wall skin-friction distribution are shown in Fig. 3. The results for the two formulations are in excellent agreement and correlate well with Inoue's results, except for the outflow conditions. In Fig. 3 the solution for the higher Reynolds number, $R = 12500$, with a larger separation region is also presented.

The biharmonic equation is the most efficient of the three formulations in terms of CPU time and storage requirements. The biharmonic program is more than two-times faster and requires more than a factor two less memory than the stream-function vorticity program. The primitive variable method is the slowest among the three formulations and it puts severe demands on the computer storage requirements. In this case the bandwidth is $0(6N)$ and this coupled with the increase in the number of variables results in more than a four-fold increase in storage and CPU time as compared to the biharmonic program.

Conclusions

Three formulations of the two-dimensional Navier-Stokes equations are solved numerically using Newton's method and a direct solution routine for banded matrices. The fully implicit solution techniques use second-order central differencing for all the terms and are shown to be reliable and to provide quadratic convergence. The biharmonic formulation is most efficient in terms of CPU time and memory without loss of accuracy.

Finally, while it is well known that iterative methods (line overrelaxation or ADI) for biharmonic equations have very slow rates of convergence, the present study, using direct solvers, indicates that the biharmonic formulation is the most recommended.

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Acceleration of Iterative Algorithms for Euler Equations of Gasdynamics

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Introduction

ONE of the successful, explicit methods used to solve Euler and Navier-Stokes equations governing compressible flows is the finite-volume, Runge-Kutta, time-stepping algorithm.[1] Several attempts have been made to accelerate the iterative convergence of this method. These acceleration methods are based on local time stepping,[1] implicit residual smoothing,[1] enthalpy damping,[1] and multigrid techniques.[2] Also, an extrapolation procedure based on the power method and the Minimal Residual Method (MRM) were applied[2] to the Jameson's multigrid algorithm. The MRM has not been shown to accelerate the scheme without multigriding. It uses same values of optimal weights for the corrections to every equation in a system. If each component of the solution vector in a system of equations is allowed to have its own convergence speed, then a separate sequence of optimal weights could be assigned to each equation. This idea is the essence of the Distributed Minimal Residual (DMR) method,[3] which is based on the General Nonlinear Minimal Residual (GNLMR) concept.[4]

Time-Dependent Euler Equations

The system of time-dependent Euler equations of gasdynamics in a two-dimensional space can be written in a general conservative form[1] as

$$\frac{\partial Q}{\partial \tau} + \frac{\partial E}{\partial \xi} + \frac{\partial F}{\partial \eta} = 0 \qquad (1)$$

where the global solution vectors combining mass, ξ-momentum, η-momentum, and energy conservation equations are

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